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Tensor Product of Difference Posets and Effect Algebras

Anatolij Dvurečenskij[1]

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A tensor product of difference posets and/or, equivalently, of effect algebras, which generalize orthoalgebras and orthomodular posets, is defined, and an equivalent condition is presented. The proof uses the notion of D-test spaces generalizing test spaces of Randall and Foulis. In particular, we show that a tensor product for difference posets with a nonempty system of probability measures exists.

1. INTRODUCTION

The event structure of a quantum physical system is identified with a quantum logic (Busch *et al.*, 1991) or an orthoalgebra (Randall and Foulis, 1981; Foulis *et al.*, 1992), in contrast to classical mechanics, where it is assumed to be a Boolean algebra. An important problem is that of coupled systems of two independent physical systems P and Q. The event structure L of this coupled system L, if it exists, is usually called a tensor product, and we write $L = P \otimes Q$.

Tensor products in various approaches have been studied (Sikorski, 1964; Aerts and Daubechies, 1978; Matolesi, 1975; Foulis, 1989; Foulis and Randall, 1981; Kläy *et al.*, 1987; Lock, 1990; Pulmannová, 1985; Randall and Foulis, 1981; Wilce, 1990; Zecca, 1978; Hudson and Pulmannová, 1993). One of the factors motivating the study of orthoalgebras was the discovery (Foulis and Randall, 1981) that there does not exist a satisfactory theory of tensor products of orthomodular posets and lattices. A tensor product of orthoalgebras has been investigated by Foulis and Bennett (1993) via a universal mapping property, and a tensor product of an orthoalgebra and a

[1]Mathematical Institute, Slovak Academy of Sciences, SK-814 73 Bratislava, Slovakia. E-mail: dvurecen@mau.savba.sk.

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Boolean algebra is given in Foulis and Pták (1993). Unfortunately, Foulis and Bennett (1993) showed that the category of orthoalgebras is not closed with respect to the formation of tensor products. Therefore, more available structures have been expected.

At the beginning of 1990s, my former student, Kôpka and Chovanec, have presented a new axiomatic model, *difference posets*. Their idea is simple: If we have two comparable events a and b ($a \leq b$), then our knowledge on a and b entails the complete knowledge of the rest of a in b, i.e., $b \ominus a$. First this idea was applied to fuzzy set ideas in quantum mechanics (Kôpka, 1992) and then presented in a general algebraic form (Kôpka and Chovanec, 1994). Difference posets generalize quantum logics, orthoalgebras, as well as the set of all effects [i.e., the system of all Hermitian operators A on a Hilbert space H with $O \leq A \leq I$, which are important for modeling unsharp measurement in a Hilbert space quantum mechanics (Busch *et al.*, 1991)]. A big advantage of difference posets is the possibility of handling self-orthogonal events.

The difference posets with a primary notion of the difference of comparable events have been very well adopted by the Slovak group as well as others. Recently Pulmannová (1994), Foulis and Bennett (1994), Gudder (1994), and Greechie (1994) studied a structure, called now an *effect algebra*, with a primary operation $\oplus$ slightly generalizing orthoalgebras. Here $a \oplus b$ is defined only for mutually excluded events a and b. But, as stressed, difference posets and effect algebras are practically the same thing, because $\oplus$ can be uniquely derived from $\ominus$ to be an effect algebra and vice versa.

This spring, my friend, R. Giuntini, called my attention to the fact that in 1989, he and Greuling, starting from orthoalgebras, presented a structure, a *weak orthoalgebra*, with a primary operation $\oplus$, which is perfectly the same thing as an effect algebra, and now the circle is closed.

Today difference posets or effect algebras present an interesting structure having many possibilities, from quantum structures to Abelian semigroups or groups (Foulis and Bennett, 1994; Wilce, 1994),[2] and, as Shakespeare said, a rose is beautiful under any name. I am very glad that the interest of the mathematics and physics community in these structures has risen with the appearance of Kôpka and Chovanec's model of difference posets.

The aim of the present paper is to describe the situation of a tensor product in the category of difference posets, or equivalently, in the category of effect algebras. The first result was given in Dvurečenskij (1994). Here we present a new proof using the notion of D-test spaces developed in

[2]For example, in Foulis and Bennett (1994) it is shown that, for any D-poset L, there exists a universal group representation ($\mathcal{G}$, γ), where $\mathcal{G}$ is an Abelian group and γ is a $\mathcal{G}$-valued measure on L. We recall that the trivial case $\mathcal{G} = \{0\}$ is not excluded.

Dvurečenskij and Pulmannová (1994) and compare it with another approach to this problem.

2. DIFFERENCE POSETS

A D-*poset*, or a *difference poset*, is a partially ordered set[3] L with a partial ordering $\leq$, greatest element 1, and partial binary operation $\ominus$: $L \times L \to L$, called a *difference*, such that, for $a, b \in L$, $b \ominus a$ is defined if and only if $a \leq b$, and such that the following axioms hold for $a, b, c \in L$:

(DPi) $b \ominus a \leq b$.
(DPii) $b \ominus (b \ominus a) = a$.
(DPiii) $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The following statements have been proved in Kôpka and Chovanec (1994).

Proposition 2.1. Let a, b, c, d be elements of a D-poset L. Then:

(i) $1 \ominus 1$ is the smallest element of L; denote it by 0.
(ii) $a \ominus 0 = a$.
(iii) $a \ominus a = 0$.
(iv) $a \leq b \Rightarrow b \ominus a = 0 \Leftrightarrow b = a$.
(v) $a \leq b \Rightarrow b \ominus a = b \Leftrightarrow a = 0$.
(vi) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(vii) $b \leq c, a \leq c \ominus b \Rightarrow b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.
(viii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Example 2.2. The set $\mathscr{E}(H)$ of all Hermitian operators A on H such that $O \leq A \leq I$, where I is the identity operator on H, is a difference poset (which is not an orthoalgebra); a partial ordering $\leq$ is defined via $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = B \ominus A$ iff $(Ax, x) - (Bx, x) = (Cx, x)$, $x \in H$.

Example 2.3. Let the closed interval [0, 1] be ordered by the natural ordering. Let g be any continuous, increasing mapping from [0, 1] onto [0, 1] such that $g(0) = 0$ and $g(1) = 1$ (called a generator). Define a partial binary operation $\ominus_g$ via

$$b \ominus_g a := g^{-1}(g(b) - g(a)) \tag{2.1}$$

Then L with $\leq$, 1, and $\ominus_g$ is a D-poset (Kôpka and Chovanec, 1994). In particular, if $g = \mathrm{id}_{[0,1]}$, then $b \ominus_{\mathrm{id}} a = b - a$.

[3] We assume that card $L > 1$.

Conversely, by Mesiar (1994), any difference $\ominus$ on the D-poset [0, 1] is equal to some $\ominus_g$ defined by (2.1).

For any element $a \in L$, we put

$$a^{\perp} := 1 \ominus a$$

Then (i) $a^{\perp\perp} = a$; (ii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$. Two elements a and b of L are *orthogonal*, and we write $a \perp b$ iff $a \leq b^{\perp}$ (iff $b \leq a^{\perp}$).

Now we introduce a partial binary operation $\oplus$: $L \times L \to L$ such that an element $c = a \oplus b$ in L is defined iff $a \perp b$, and for c we have

$$c = a \oplus b = (a^{\perp} \ominus b)^{\perp} = (b^{\perp} \ominus a)^{\perp} \tag{2.2}$$

The operation $\oplus$ is commutative and associative.

3. EFFECT ALGEBRAS

An *effect algebra* (Greechie, 1994) is a set L with two particular elements 0, 1, and with a partial binary operation $\oplus$: $L \times L \to L$ such that for all a, b, $c \in L$ we have:

(EAi) If $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity).

(EAii) If $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).

(EAiii) For any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation).

(EAiv) If $1 \oplus a$ is defined, then $a = 0$ (zero–one law).

If the assumptions of (EAii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

If (EAiv) is changed to

(OA) If $a \oplus a$ is defined, then $a = 0$ (consistency)

we say that L is an *orthoalgebra*.

Let a and b be two elements of an effect algebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L, (ii) a is *less than or equal to* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$), and (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$ and it is written as $a^{\perp}$.

If $a \leq b$, for the element c in (ii) with $a \oplus c = b$ we write $c = b \ominus a$, and c is called the *difference* of a and b. It is evident that

$$b \ominus a = (a \oplus b^{\perp})^{\perp} \tag{3.1}$$

It is clear that if $(L, 1, \leq, \ominus)$ is a difference poset, then $(L, 0, 1, \oplus)$ is an effect algebra, where $\oplus$ is defined by (2.1) and $0 = 1 \ominus 1$. Conversely, if $(L, 0, 1, \oplus)$ is an effect algebra, then $(L, 1, \leq, \ominus)$ is a difference poset, where $\ominus$ is defined by (3.1) and $\leq$ is naturally defined in L. In other words, difference posets and effect algebras are the same thing.

Let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in L. Recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n \tag{3.2}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in D-posets we conclude that (3.2) is correctly defined. By definition we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n} \tag{3.3}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n) \tag{3.4}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ in L is *$\oplus$-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a *$\oplus$-sum*, $\oplus_{i=1}^{n} a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n \tag{3.5}$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\oplus$-orthogonal.

Let n be a nonnegative integer and let $a \in L$. If $n = 0$, we define $na := 0$, and if $n = 1$, we define $na := a$. If $n > 1$, let $a_i := a$ for $i = 1, 2, \ldots, n$. We say that na is defined iff $\{a_1, \ldots, a_n\}$ is $\oplus$-orthogonal, in which case we define $na := \oplus_{i=1}^{n} a_i$.

A finite *decomposition* of 1 is any $\oplus$-orthogonal finite sequence $(a_1, \ldots, a_n)$ such that $\oplus_{i=1}^{n} a_i = 1$.

4. D-TEST SPACES

For orthoalgebras, a test space is a basic notion developed by Randall and Foulis (1981) because orthoalgebras and algebraic test spaces are in an intimate correspondence (Foulis *et al.*, 1992). In this section, we give a generalization of test spaces, D-test space, which was originally presented in Dvurečenskij and Pulmannová (1994); here we give a more elementary definition. D-test spaces will be used for a tensor product of difference posets or, equivalently, of effect algebras. In fact, it will deal with so-called finite D-test spaces, which are a special case of general D-tests from Dvurečenskij and Pulmannová (1994); however, for our aims it is satisfactory only to have this type of D-test space.

Let X be a nonempty set; elements of X are called *outcomes*. For two finite sequences $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_m)$ from $[X] := \bigcup_{n=0}^{\infty} X^n$,[4] we write $F \preceq G$ iff $n \leq m$ and if there is an injection σ: $\{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $F = G \circ \sigma$, i.e., $f_i = g_{\sigma(i)}$ for any $i = 1, \ldots, n$. If, for $F, G \in [X]$ we have $F \preceq G$ and $G \preceq F$, we say that F and G are *equivalent*, in symbols $F \sim G$, and $\sim$ is an equivalence relation on $[X]$. In what follows, we shall identify finite sequences which are equivalent.

Therefore, we can define unambiguously a sequence $F \dot{\cup} G$, when $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_m)$, via

$$F \dot{\cup} G = (f_1, \ldots, f_n, g_1, \ldots, g_m)$$

By $\mathcal{R}(F)$ we denote the range of the sequence $F = (a_1, \ldots, a_n)$, i.e., $\mathcal{R}(F) = \{a_1, \ldots, a_n\}$.

Definition 4.1. Let $\mathcal{T} \subseteq [X]$ be nonempty, where $X \neq \emptyset$. We say that the pair $(X, \mathcal{T})$ is a D-*test space* iff the following two conditions are satisfied:

(i) For every $x \in X$ there is an $T \in \mathcal{T}$ such that $x \in \mathcal{R}(T)$.

(ii) If $S, T \in \mathcal{T}$ and $S \preceq T$, then $S \sim T$.

Any element of $\mathcal{T}$ is said to be a D-*test*.

We recall that if, for any D-test $T = (a_1, \ldots, a_n) \in \mathcal{T}$ we have $\mathrm{card}\{a_1, \ldots, a_n\} = n$, then $(X, \mathcal{T})$ is a test space in the sense of Randall and Foulis (1981).

Lemma 4.2. If $F \in [X]$ is a D-test, then $F \in \bigcup_{n=1}^{\infty} X^n$.

Proof. Let $F = \emptyset \in \mathcal{T}$. Then, for any $T \in \mathcal{T}$, $F \preceq T$ implies $F \sim T$ and, by (ii) of Definition 4.1, $T = \emptyset$, which is in contradiction with the condition (i) of Definition 4.1. ■

Definition 4.3. Let $(X, \mathcal{T})$ be a D-test space. We say that $G \in [X]$ is an *event* iff there is a D-test $T \in \mathcal{T}$ such that $G \preceq T$. Let us denote the set of all events in $\mathcal{T}$ by $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$.

Clearly, $\emptyset \in \mathcal{E}$.

Definition 4.4. Let $(X, \mathcal{T})$ be a D-test space. We say that two events F and G are (i) *orthogonal* to each other, in symbols $F \perp G$, iff there is a D-test $T \in \mathcal{T}$ such that $F \dot{\cup} G \preceq T$; (ii) *local complements* of each other, in

[4] If $n = 0$, then $x^0 := \{\emptyset\}$.

symbols F loc G, iff there is a D-test $T \in \mathcal{T}$ such that $F \dot{\cup} G \sim T$; (iii) *perspective with axis* H iff they share a common local complement H. We write $F \approx_H G$ or $F \approx G$ if the axis is not emphasized.

The D-test space $(X, \mathcal{T})$ is D-*algebraic* iff, for $F, G, H \in \mathcal{E}$, $F \approx G$ and $F \perp H$ entail $G \perp H$. For simplicity, we usually refer to X, rather than to $(X, \mathcal{T})$, as a D-test space; similarly, we put $\mathcal{E} = \mathcal{E}(X)$. In what follows, let X be a D-test space.

It is possible to show that if X is D-algebraic, then $\approx$ is an equivalence relation on $\mathcal{E}$, and we can define, for $F \in \mathcal{E}$, $\pi(F) := \{G \in \mathcal{E}: G \approx F\}$ and we refer to $\pi(F)$ as the *proposition affiliated with* F. The set

$$\Pi = \Pi(X) := \{\pi(F): F \in \mathcal{E}\} \tag{4.1}$$

is called the *logic* of the D-test space X.

We define $0, 1 \in \Pi$ by

$$0 = \pi(\emptyset), \qquad 1 = \pi(T) \tag{4.2}$$

where T is any D-test.

Similarly, we define, for $F \in \mathcal{E}$, the negation $\pi(F)'$ of the proposition $\pi(F) \in \Pi(X)$ by $\pi(F)' = \pi(G)$, whenever G is any local complement of F in $\mathcal{E}$.

Theorem 4.5. Let X be a D-algebraic D-test space. Then the logic $\Pi(X)$ of X can be organized into a D-poset or, equivalently, into an effect algebra.

Proof. For two events $a, b \in \Pi(X)$, we define $a \leq b$ iff there are $F, G, H \in \mathcal{E}$ such that $a = \pi(F)$, $b = \pi(G)$, $H \perp F$, and $G \approx F \dot{\cup} H$. Moreover, we define $b \ominus a$ via $b \ominus a := \pi(H)$. Then $\leq$ is a partial ordering on $\Pi(X)$ with the smallest element $\pi(\emptyset)$ and the greatest element $\pi(T)$, where T is any D-test. A direct calculation shows that $\Pi(X)$ with $\leq$, $\pi(1)$, $\ominus$ is a difference poset in question. ∎

Let P and L be two D-posets. A mapping $\phi: P \to L$ is said to be:

(i) a *morphism* iff $\phi(1) = 1$, and $p \perp q$, $p, q \in P$, implies $\phi(p) \perp \phi(q)$ and $\phi(p \oplus q) = \phi(p) \oplus \phi(q)$;
(ii) a *monomorphism* iff ϕ is a morphism and $\phi(p) \perp \phi(q)$ iff $p \perp q$;
(iii) an *isomorphism* iff ϕ is a surjective monomorphism, and we say that P is isomorphic to L;
(iv) a *state* if L is a D-poset from Example 2.3.

Let P, Q, L be D-posets. A mapping $\beta: P \times Q \to L$ is called a *bimorphism* iff:

(i) $a, b \in P$ with $a \perp b$, $q \in Q$ imply $\beta(a, q) \perp \beta(b, q)$ and $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$.

(ii) $c, d \in Q$ with $c \perp d$, $p \in P$ imply $\beta(p, c) \perp \beta(p, d)$ and $\beta(p, c \oplus d) = \beta(p, c) \oplus \beta(p, d)$.
(iii) $\beta(1, 1) = 1$.

If $\beta: P \times Q \to L$ is a bimorphism, then $\beta(\cdot, 1): P \to L$ and $\beta(1, \cdot): Q \to L$ are morphisms. Therefore, for $p \in P$ and $q \in Q$, we have $\beta(p, 1)^\perp = \beta(p^\perp, 1)$, $\beta(1, q)^\perp = \beta(1, q^\perp)$, and $\beta(p, 0) = \beta(0, q) = 0$.

Also, if $a, b, p \in P$ and $c, d, q \in Q$, we have $a \le b \Rightarrow \beta(a, q) \le \beta(b, q)$ and $c \le d \Rightarrow \beta(p, c) \le \beta(p, d)$.

Theorem 4.6. Let L be a D-poset. Then there exists a D-algebraic D-test space $(X, \mathcal{T})$ such that L is isomorphic to $\Pi(X)$.

Proof. Let $X = L\setminus\{0\}$ and let $\mathcal{T}$ be the set of all finite partitions of 1 in L consisting of nonzero elements. It is possible to show that $(X, \mathcal{T})$ is a D-algebraic D-test space.

Let $\Pi(X)$ be the logic of X. By Theorem 4.5, $\Pi(X)$ is a D-poset. Define a mapping $\phi: L \to \Pi(X)$ by

$$\phi(a) = \begin{cases} \pi(\{a\}) & \text{if } a \neq 0 \\ \pi(\emptyset) & \text{if } a = 0 \end{cases}$$

The mapping ϕ is an isomorphism from L onto $\Pi(X)$. ■

5. TENSOR PRODUCTS

In the present section, we define a tensor product of difference posets and a necessary and sufficient condition for it to exist. The following definition was presented originally in Dvurečenskij (1994).

Definition 5.1. Let P and Q be difference posets. We say that a pair (T, τ) consisting of a difference poset T and a bimorphism $\tau: P \times Q \to T$ is a *tensor product* of P and Q iff the following conditions are satisfied:

(i) If L is a D-poset and $\beta: P \times Q \to L$ is a bimorphism, there exists a morphism $\phi: T \to L$ such that $\beta = \phi \circ \tau$.
(ii) Every element of T is a finite orthogonal sum of elements of the form $\tau(p, q)$ with $p \in P$, $q \in Q$.

It is not hard to show that if a tensor product (T, τ) of P and Q exists, it is unique up to an isomorphism, i.e., if (T, τ) and (T^*, τ^*) are tensor products of D-posets P and Q, then there is a unique isomorphism $\phi: T \to T^*$ such that $\phi(\tau(p, q)) = \tau^*(p, q)$ for all $p \in P$, $q \in Q$.

Now we present the main assertion (see also Dvurečenskij, 1994) of this section using the notion of D-test spaces.

Theorem 5.2. The difference posets P and Q admit a tensor product if and only if there is at least one difference poset L for which there is a bimorphism $\beta: P \times Q \to L$.

Proof. The necessary condition is evident.

For the sufficiency, suppose that N is the subset of $P \times Q$ consisting of all (p, q) such that $\beta(p, q) = 0$ for every bimorphism β on $P \times Q$. Define $X := (P \times Q)\backslash N$. If $A = \{(p_i, q_i)\}_{i=1}^{n}$ is a finite sequence of elements from $P \times Q$ and $\beta: P \times Q \to L$ is a bimorphism, it is clear that $\beta(A)$ is $\oplus$-orthogonal iff $\beta(\tilde{A})$ is $\oplus$-orthogonal, where $\tilde{A} = \{(p_i, q_i)\}_{i=1}^{m}$, $0 \le m \le n$, and $(p_i, q_i) \in A$, $(p_i, q_i) \in X$; in this case $\oplus\ \beta(A) = \oplus\ \beta(A)$ for every bimorphism β on $P \times Q$.

Denote by $\mathcal{H}$ the set of all finite sequences H of elements from X such that for every bimorphism β, $\beta(H)$ is a finite decomposition of 1. It is clear that $\mathcal{H}$ is nonempty, since $\{(1, 1)\} \in \mathcal{H}$. We assert that that $(X, \mathcal{H})$ is an algebraic D-test space. Then the set of all effects, $\mathcal{E}(\mathcal{H})$, is the set of all finite sequences $A = \{(p_i, q_i)\}_{i=1}^{n}$ (may also be empty) such that there is a system $\{(a_j, b_j)\}_{j=1}^{m}$ of elements from X such that $((p_1, q_1), \ldots, (p_n, q_n), (a_1, b_1), \ldots, (a_m, b_m)) \in \mathcal{H}$. According to Theorem 4.5, $\Pi(X) := \{\pi(A): A \in \mathcal{E}(\mathcal{H})\}$ can be organized into a difference poset.

Now put $P \otimes Q := \Pi(X)$ and define a mapping $\otimes: P \times Q \to P \otimes Q$ via

$$\otimes(p, q) = \begin{cases} \pi(\{(p, q)\}), & (p, q) \in X \\ 0, & (p, q) \notin X \end{cases}$$

For simplicity, we often write $p \otimes q$ rather than $\otimes(p, q)$.

We assert that $\otimes: P \times Q \to P \otimes Q$ is a bimorphism. Indeed, since $\{(1, 1)\} \in \mathcal{H}$, we have $\otimes(1, 1) = \pi(\{(1, 1)\}) = 1$. Suppose that $a, b \in P$ with $a \perp b$ and $q \in Q$. We have to show that $a \otimes q \perp b \otimes q$ and $(a \oplus b) \otimes q = (a \otimes q) \oplus (b \otimes q)$. If $(a, q) \in N$ or $(b, q) \in N$, this is clear, so we may assume that $(a, q), (b, q) \in X$. If β is any bimorphism on $P \times Q$, we have $\beta(a \oplus b, q) = \beta(a, q) \oplus \beta(b, q)$. Hence $\{(a \oplus b, q)\} \sim \{(a, q), (b, q)\}$, so that $(a \oplus b) \otimes q = (a \otimes q) \oplus (b \otimes q)$.

A similar argument shows that $p \otimes (c \oplus d) = (p \otimes c) \oplus (p \otimes d)$ holds for $p \in P$ and $c, d \in Q$ with $c \perp d$.

It remains to prove that $(P \otimes Q, \otimes)$ is a tensor product of P and Q. Since every element of $P \otimes Q = \Pi(X)$ can be written in the form $\pi(A) = \oplus\{\pi(\{(p, q)\}): (p, q) \in A\} = \oplus\{p \otimes q: (p, q) \in A\}$, every element of $P \otimes Q$ is a $\oplus$-sum of finitely many elements $p \otimes q$.

Finally, suppose that $\beta: P \times Q \to L$ is a bimorphism. If $A, B \in \mathcal{E}(\mathcal{H})$ and $A \sim B$, then $\oplus\ \beta(A) = \oplus\ \beta(B)$, hence we can define a mapping $\phi: P \otimes Q \to L$ by $\phi(\pi(A)) = \oplus\ \beta(A)$ for every $\pi(A) \in \Pi(X)$. Obviously, ϕ is a morphism and we have $\beta(p, q) = \phi(p \otimes q)$ for all $p \in P$, $q \in Q$. ■

Unless confusion threatens, we usually refer to $P \otimes Q$ rather than to $(P \otimes Q, \otimes)$ as being a tensor product.

Theorem 5.3. Let both D-posets P and Q possess at least one state. Then the tensor product of P and Q exists in the category of D-posets. In addition, for any state μ on P and any state ν on Q there is a unique state $\mu \otimes \nu$ on $P \otimes Q$ such that

$$\mu \otimes \nu(p \otimes q) = \mu(p)\nu(q), \qquad p \in P, \quad q \in Q \tag{5.1}$$

Proof. Let $L = [0, 1]$ be endowed with the natural ordering and the difference $b \ominus a := b - a$, $a, b \in [0, 1]$. Then L is a D-poset. Choose two states μ and ν on P and Q, respectively, and define a mapping $\beta_{\mu\nu}$: $P \times Q \to L$ such that

$$\beta_{\mu\nu}(p, q) = \mu(p)\cdot\nu(q), \qquad p \in P, \quad q \in Q \tag{5.2}$$

Then $\beta_{\mu\nu}$ is a bimorphism, which, by Theorem 5.2 is a necessary and sufficient condition for P and Q to admit a tensor product.

Since $\beta_{\mu\nu}$ is a bimorphism, from the definition of $P \otimes Q$ it follows that there is a morphism ϕ: $P \otimes Q \to [0, 1]$ such that $\phi(p \otimes q) = \beta_{\mu\nu}(p, q)$, $p \in P$, $q \in Q$. But this means that ϕ is a state on $P \times Q$ with the desired property (5.1). The uniqueness of ϕ is clear due to the property of $P \otimes Q$ that any element $t \in P \otimes Q$ is of the form $t = \oplus_{i=1}^{n} p_i \otimes q_i$. ∎

Foulis and Bennett (1993) showed that the orthoalgebra, the Fano plane, illustrated by the Greechie diagram in Fig. 1, has no tensor product $F \otimes F$ in the category of orthoalgebras. Since on F there is a unique state, say μ, defined by $\mu(x) = 1/3$, $x \in \{a, b, c, d, e, f, g\}$, using Theorem 5.3, we see that $F \otimes F$ as a D-poset exists.

Therefore, in Dvurečenskij (1994) the problem of whether any two D-posets or, equivalently, any two effect algebras, admit a tensor product has been formulated. Recently at the 1994 IQSA Conference in Prague it was

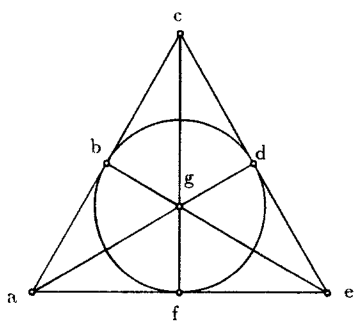

Fig. 1.

announced that a positive solution to this problem has been found, so that the category of difference posets, or, equivalently, of effect algebras is closed with respect to the operation of the tensor product. Unfortunately, the proof possesses a gap, and hence the problem remains open. Nevertheless, Theorem 5.3 ensures that for physically reasonable D-posets, i.e., those having at least one state, any two such D-posets admit a tensor product, which for orthoalgebras was not always possible; see the Fano plane.

Some questions related to the tensor product of difference posets are given in Wilce (1994), where the notion of bisummable functions is used to obtain the analogous assertion to that in Theorem 5.1.

NOTE ADDED IN PROOFS

Recently S. P. Gudder and R. Greechie: Effect algebras counterexamples, Mathematica Slovaca (to appear) found an example of an effect algebra P such that $P \otimes P$ fails.

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